

# Askey-Wilson Polynomials for Root Systems of Type BC

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ABSTRACT. This paper introduces a family of Askey-Wilson type orthogonal polynomials in  $n$  variables associated with a root system of type  $BC_n$ . The family depends, apart from  $q$ , on 5 parameters. For  $n = 1$  it specializes to the four-parameter family of one-variable Askey-Wilson polynomials. For any  $n$  it contains Macdonald's two three-parameter families of orthogonal polynomials associated with a root system of type  $BC_n$  as special cases.

## 1. Introduction

In recent years, some families of orthogonal polynomials associated with root systems were introduced. The families studied by Heckman & Opdam [6], [4], [5] become Jacobi polynomials for root system  $BC_1$ . The families studied by Macdonald (see [11] for root system  $A_n$  and [12] for general root systems) become continuous  $q$ -ultraspherical polynomials for root system  $A_1$  and continuous  $q$ -Jacobi polynomials for root system  $BC_1$  (see Askey & Wilson [1, §4]). For all root systems Macdonald's polynomials tend to the Heckman-Opdam polynomials as  $q$  tends to 1.

This paper introduces a family of Askey-Wilson type polynomials for root system  $BC_n$  which depends, apart from  $q$ , on 5 parameters. For  $n = 1$  it specializes to the four-parameter family of Askey-Wilson polynomials. For any  $n$  it contains Macdonald's two three-parameter families as special cases: for the pair  $(BC_n, B_n)$  directly and for the pair  $(BC_n, C_n)$  when  $q$  is replaced by  $q^2$ . Moreover, the weight function integrated over the orthogonality domain was explicitly evaluated by Gustafson [3] as a generalization of Selberg's beta integral.

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The proofs in this paper are very much inspired by Macdonald's proofs in [12], in particular by his proofs in case of root systems  $E_8$ ,  $F_4$ ,  $G_2$ , where there is no minuscule fundamental weight available.

The contents of this paper are as follows. Section 2 summarizes Macdonald's results. The special case  $BC_n$  of these results is discussed in §3 and the further specialization to  $BC_1$  in §4. The long section 5 introduces Askey-Wilson polynomials for root system  $BC_n$ , shows that these polynomials are eigenfunctions of a certain difference operator and establishes the full orthogonality of the polynomials. Finally, in §6, special cases and open problems are discussed

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## 2. Summary of Macdonald's results

In this section we summarize Macdonald's [12] results on orthogonal polynomials associated with root systems. See Humphreys [7] and Bourbaki [2, Chap.6] for preliminaries on root systems. Let  $V$  be a finite dimensional real vector space with inner product  $(\cdot, \cdot)$ . Write  $|v| := \langle v, v \rangle^{1/2}$  for the norm of  $v \in V$ . Write

$$v^\vee := 2v/|v|^2, \quad 0 \neq v \in V.$$

Let  $R$  be a not necessarily reduced root system spanning  $V$ . Let  $S$  be a reduced root system in  $V$  such that the set of lines  $\{\mathbb{R}\alpha \mid \alpha \in R\}$  equals  $\{\mathbb{R}\alpha \mid \alpha \in S\}$ . Then the pair  $(R, S)$  is called *admissible* and  $R$  and  $S$  have the same Weyl group  $W$ . Now, for each  $\alpha \in R$ , there is a (unique)  $u_\alpha > 0$  such that  $\alpha_* := u_\alpha^{-1}\alpha \in S$ . Assume that  $R$  is irreducible. It can be arranged, after possible dilation of  $R$  and  $S$ , that  $u_\alpha$  takes values in  $\{1, 2\}$  or in  $\{1, 3\}$ .

Let  $0 < q < 1$ . Put  $q_\alpha := q^{u_\alpha}$ . Let  $\alpha \mapsto t_\alpha$  be a  $W$ -invariant function on  $R$ , taking values in  $(0, 1)$  (for convenience). Then  $t_\alpha$  only depends on  $|\alpha|$ . Put  $t_\alpha := 1$  if  $\alpha \in V \setminus R$ . Let  $k_\alpha \geq 0$  be such that  $q_\alpha^{k_\alpha} = t_\alpha$ .

Let  $R^+$  be a choice for the set of positive roots in  $R$ . Let

$$Q := \mathbb{Z}\text{-Span}(R), \quad Q^+ := \mathbb{Z}_+\text{-Span}(R^+).$$

Here, and throughout the paper,  $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ . Let

$$P := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall \alpha \in R\}, \quad P^+ := \{\lambda \in V \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_+ \quad \forall \alpha \in R^+\}$$

be respectively the weight lattice of  $R$  and the cone of dominant weights. Define a partial order on  $P$  by  $\lambda \geq \mu$  iff  $\lambda - \mu \in Q^+$ .

For  $\lambda \in P$  let  $e^\lambda$  be the function on  $V$  defined by

$$e^\lambda(x) := e^{i\langle \lambda, x \rangle}, \quad x \in V.$$

Extend this holomorphically to  $V + iV$ . If  $f$  is a function on  $V$  then put  $(wf)(x) := f(w^{-1}x)$  for  $w \in W, x \in V$ . Hence  $we^\lambda = e^{w\lambda}$ . Let  $A$  be the complex linear span of the  $e^\lambda$  ( $\lambda \in P$ ). Let  $A^W$  denote the space of  $W$ -invariants of  $A$ . Put

$$m_\lambda := |W_\lambda|^{-1} \sum_{w \in W} e^{w\lambda} = \sum_{\mu \in W\lambda} e^\mu, \quad \lambda \in P^+.$$

Here  $W_\lambda$  denotes the stabilizer of  $\lambda$  in  $W$ . The  $m_\lambda$  ( $\lambda \in P^+$ ) form a basis of  $A^W$ . Note that  $\overline{m_\lambda(x)} = m_\lambda(-x)$  ( $\lambda \in P^+, x \in V$ ). If  $-id \in W$  then  $f(x) = f(-x)$  for  $f \in A^W$ . In particular, we will then have that  $m_\lambda$  is real-valued on  $V$ .

Let  $R^\vee := \{\alpha^\vee \mid \alpha \in R\}$  be the root system dual to  $R$ . Let  $Q^\vee := \mathbb{Z}\text{-Span}(R^\vee)$ . Then  $T := V/(2\pi Q^\vee)$  is a torus. Let  $\dot{x}$  be the image in  $T$  of  $x \in V$ . Let  $d\dot{x}$  be the normalized Haar measure on  $T$ . For  $\lambda \in P$  the function  $\dot{x} \mapsto e^\lambda(x)$  is well-defined on  $T$ . For  $a, a_1, \dots, a_k \in \mathbb{C}$  put

$$(a; q)_\infty := \prod_{j=0}^\infty (1 - aq^j), \quad (a_1, \dots, a_k; q)_\infty := \prod_{i=1}^k (a_i; q)_\infty.$$

Define

$$\Delta := \prod_{\alpha \in R} \frac{(t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}, \quad \Delta^+ := \prod_{\alpha \in R^+} \frac{(t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}{(t_\alpha t_{2\alpha}^{1/2} e^\alpha; q_\alpha)_\infty}.$$

Then  $\Delta = \Delta^+ \overline{\Delta^+}$ . Define a hermitian inner product on  $A^W$  by

$$\langle f, g \rangle := |W|^{-1} \int_T f(\dot{x}) \overline{g(\dot{x})} \Delta(x) d\dot{x}.$$

DEFINITION 2.1. For  $\lambda \in P^+$  let  $P_\lambda \in A^W$  be characterized by the two conditions

- (i)  $P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda, \mu} m_\mu$  for certain complex coefficients  $u_{\lambda, \mu}$ ;
- (ii)  $\langle P_\lambda, m_\mu \rangle = 0$  if  $\mu < \lambda$ .

THEOREM 2.2.

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{if } \lambda \neq \mu.$$

Define

$$(T_v f)(x) := f(x - i(\log q)v), \quad x, v \in V,$$

for functions  $f$  being analytic on a suitable subset of  $V + iV$  containing  $V$ . Hence

$$T_v e^\lambda = q^{\langle v, \lambda \rangle} e^\lambda, \quad \lambda \in P.$$

Let  $\sigma \in V$  be such that  $\langle \sigma, \alpha_* \rangle$  takes just two values 0 and 1 as  $\alpha$  runs through  $R^+$  ( $\sigma$  is a so-called minuscule fundamental weight for  $S^\vee$ ). Such  $\sigma$  exists for all

$S$  not being of type  $E_8$ ,  $F_4$  or  $G_2$ . In these last three cases we can choose  $\sigma$  such that  $\langle \sigma, \alpha_* \rangle$  takes values 0, 1 and 2 as  $\alpha$  runs through  $R^+$ . Now put

$$\begin{aligned}\Phi_\sigma &:= \frac{T_\sigma \Delta^+}{\Delta^+}, \\ E_\sigma f &:= |W_\sigma|^{-1} \sum_{w \in W} w(\Phi_\sigma T_\sigma f), \\ D_\sigma f &:= |W_\sigma|^{-1} \sum_{w \in W} w(\Phi_\sigma(T_\sigma f - f)), \\ \tilde{m}_\sigma(\lambda) &:= |W_\sigma|^{-1} \sum_{w \in W} q^{(w\sigma, \lambda)}, \\ \rho_k &:= \frac{1}{2} \sum_{\alpha \in R^+} k_\alpha \alpha.\end{aligned}$$

**THEOREM 2.3.**  $D_\sigma$  maps  $A^W$  into itself. The  $P_\lambda$  are eigenfunctions of  $D_\sigma$  with eigenvalue

$$q^{(\sigma, \rho_k)} (\tilde{m}_\sigma(\lambda + \rho_k) - \tilde{m}_\sigma(\rho_k)).$$

If  $S$  is not of type  $E_8$ ,  $F_4$  or  $G_2$  then  $E_\sigma$  maps  $A^W$  into itself, the  $P_\lambda$  are also eigenfunctions of  $E_\sigma$  with eigenvalue

$$q^{(\sigma, \rho_k)} \tilde{m}_\sigma(\lambda + \rho_k).$$

and

$$D_\sigma = E_\sigma - E_\sigma(1)$$

with  $E_\sigma(1)$  scalar.

### 3. The case $R = BC_n$

Identify  $V$  with  $\mathbb{R}^n$  and let  $\varepsilon_1, \dots, \varepsilon_n$  be its standard basis. Consider in  $V$  the root systems

$$R := \{\pm\varepsilon_j\} \cup \{\pm 2\varepsilon_j\} \cup \{\pm\varepsilon_i \pm \varepsilon_j\}_{i < j} \quad \text{of type } BC_n,$$

$$S_B := \{\pm\varepsilon_j\} \cup \{\pm\varepsilon_i \pm \varepsilon_j\}_{i < j} \quad \text{of type } B_n.$$

$$S_C := \{\pm\varepsilon_j\} \cup \{\frac{1}{2}(\pm\varepsilon_i \pm \varepsilon_j)\}_{i < j} \quad \text{of type } C_n.$$

Then  $S_B$  and  $S_C$  are reduced,  $R$ ,  $S_B$  and  $S_C$  have the same Weyl groups and in the mappings  $\alpha \mapsto u_\alpha^{-1}\alpha$  of  $R$  onto  $S_B$  and onto  $S_C$ ,  $u_\alpha$  take the values 1 and 2.

Note that  $R^\vee = R$  and that the weight lattice  $P$  and the root lattice  $Q$  of  $R$  are both given by

$$P = Q = \{m_1\varepsilon_1 + \dots + m_n\varepsilon_n \mid m_1, \dots, m_n \in \mathbb{Z}\}.$$

Take

$$R^+ := \{\varepsilon_j\} \cup \{2\varepsilon_j\} \cup \{\varepsilon_i \pm \varepsilon_j\}_{i < j}.$$

Then

$$P^+ = \{m_1\varepsilon_1 + \dots + m_n\varepsilon_n \mid m_1 \geq m_2 \geq \dots \geq m_n \geq 0, \quad m_1, \dots, m_n \in \mathbb{Z}\},$$

$$Q^+ = \{m_1(\varepsilon_1 - \varepsilon_2) + \dots + m_{n-1}(\varepsilon_{n-1} - \varepsilon_n) + m_n\varepsilon_n \mid m_1, \dots, m_n \in \mathbb{Z}_+\}.$$

The torus  $T := V/(2\pi Q^\vee)$  becomes  $\mathbb{R}^n/(2\pi\mathbb{Z}^n)$ . Recall that we have a partial ordering on  $P$  such that  $\lambda \geq \mu$  iff  $\lambda - \mu \in Q^+$ .

For the pair  $(R, S_B)$  we have

$$q_{\pm\varepsilon_j} = q, \quad q_{\pm 2\varepsilon_j} = q^2, \quad q_{\pm\varepsilon_i \pm \varepsilon_j} = q,$$

and there are three different parameters  $t_\alpha$ , which we write as

$$a := t_{\pm\varepsilon_j}, \quad b := t_{\pm 2\varepsilon_j}, \quad t := t_{\pm\varepsilon_i \pm \varepsilon_j}.$$

(Recall that  $t_\alpha = 1$  if  $\alpha \notin R$ .) Thus

$$(3.1) \quad \Delta^+ = \Delta_1^+ \Delta_2^+,$$

where

$$(3.2) \quad \Delta_1^+ := \prod_{j=1}^n \frac{(b^{1/2} e^{\varepsilon_j}; q)_\infty}{(a b^{1/2} e^{\varepsilon_j}; q)_\infty} \frac{(e^{2\varepsilon_j}; q^2)_\infty}{(b e^{2\varepsilon_j}; q^2)_\infty}$$

$$= \prod_{j=1}^n \frac{(e^{2\varepsilon_j}; q)_\infty}{(q^{1/2} e^{\varepsilon_j}, -q^{1/2} e^{\varepsilon_j}, a b^{1/2} e^{\varepsilon_j}, -b^{1/2} e^{\varepsilon_j}; q)_\infty}$$

and

$$(3.3) \quad \Delta_2^+ := \prod_{\alpha=\varepsilon_i \pm \varepsilon_j; i < j} \frac{(e^\alpha; q)_\infty}{(t e^\alpha; q)_\infty}.$$

For the pair  $(R, S_C)$  we have

$$q_{\pm\varepsilon_j} = q, \quad q_{\pm 2\varepsilon_j} = q^2, \quad q_{\pm\varepsilon_i \pm \varepsilon_j} = q^2,$$

and there are three different parameters  $t_\alpha$ , which we write as

$$a := t_{\pm\varepsilon_j}, \quad b := t_{\pm 2\varepsilon_j}, \quad t := t_{\pm\varepsilon_i \pm \varepsilon_j}.$$

Thus (3.1) holds with

$$(3.4) \quad \Delta_1^+ := \prod_{j=1}^n \frac{(b^{1/2} e^{\varepsilon_j}; q)_\infty}{(a b^{1/2} e^{\varepsilon_j}; q)_\infty} \frac{(e^{2\varepsilon_j}; q^2)_\infty}{(b e^{2\varepsilon_j}; q^2)_\infty}$$

$$= \prod_{j=1}^n \frac{(e^{2\varepsilon_j}; q^2)_\infty}{(a b^{1/2} e^{\varepsilon_j}, q a b^{1/2} e^{\varepsilon_j}, -b^{1/2} e^{\varepsilon_j}, -q b^{1/2} e^{\varepsilon_j}; q^2)_\infty}$$

and

$$(3.5) \quad \Delta_2^+ := \prod_{\alpha=\varepsilon_i \pm \varepsilon_j; i < j} \frac{(e^\alpha; q^2)_\infty}{(t e^\alpha; q^2)_\infty}.$$

Since  $-id \in W$  in case of root system  $BC_n$ ,  $m_\lambda$  will be real-valued and we can read for condition (ii) of Definition 2.1 that

$$\int_T P_\lambda(x) m_\mu(x) \Delta(x) dx = 0 \quad \text{if } \mu < \lambda.$$

For the element  $\sigma$  of §2 we can take  $\varepsilon_1$  in the case  $S_B$  and  $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n$  in the case  $S_C$ . In both cases  $\sigma$  is minuscule. So Theorem 2.3 is valid with these choices of  $\sigma$ . In particular, in the case  $S_B$  the polynomial  $P_\lambda$  is eigenfunction of  $D_{\varepsilon_1}$  with eigenvalue

$$(3.6) \quad \sum_{j=1}^n (ab t^{2n-j-1} (q^{\lambda_j} - 1) + t^{j-1} (q^{-\lambda_j} - 1)).$$

The choice  $\sigma := 2\varepsilon_1$  in the case  $S_C$  would give values 0, 1 and 2 for  $\langle \sigma, \alpha_* \rangle$  as  $\alpha$  runs through  $R^+$ . It will turn out in §6.1 that, in case  $S_C$ , the  $P_\lambda$  are not only eigenfunctions of  $E_{\varepsilon_1+\dots+\varepsilon_n}$  but also of  $D_{2\varepsilon_1}$ .

#### 4. The case $R = BC_1$

For  $n = 1$  the two root systems  $S_B$  and  $S_C$  coincide and the results of §3 specialize as follows. We have  $T = \mathbb{R}/(2\pi\mathbb{Z})$ ,  $P = Q = \mathbb{Z}$ ,  $P^+ = \mathbb{Z}_+$ , the partial order on  $P$  is the ordinary total order on  $\mathbb{Z}$ ,

$$m_l(x) = \begin{cases} e^{ilx} + e^{-ilx}, & l = 1, 2, \dots, \\ 1, & l = 0. \end{cases}$$

and

$$\begin{aligned} \Delta^+(x) &= \frac{(e^{2ix}, q)_\infty}{(q^{1/2} e^{ix}, -q^{1/2} e^{ix}, a b^{1/2} e^{ix}, -b^{1/2} e^{ix}; q)_\infty} \\ &= \frac{(e^{2ix}, q^2)_\infty}{(a b^{1/2} e^{ix}, q a b^{1/2} e^{ix}, -b^{1/2} e^{ix}, -q b^{1/2} e^{ix}; q^2)_\infty}. \end{aligned}$$

The inner product for  $W$ -invariant functions  $f, g$  becomes an integral over the period of  $2\pi$ -periodic even functions, so it can be written as

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^\pi f(x) \overline{g(x)} \Delta(x) dx.$$

Askey-Wilson polynomials  $p_n(y; a, b, c, d | q)$  ( $n \in \mathbb{Z}_+$ ) are defined, up to a constant factor, as polynomials of degree  $n$  in  $y$  which satisfy the orthogonality relations

$$(4.1) \quad \int_0^\pi (p_n p_m)(\cos x; a, b, c, d | q) \left| \frac{(e^{2ix}, q)_\infty}{(a e^{ix}, b e^{ix}, c e^{ix}, d e^{ix}; q)_\infty} \right|^2 dx = 0, \quad n \neq m.$$

See Askey & Wilson [1]. Here  $a, b, c, d$  are real, or if complex, appear in complex conjugate pairs, and  $|a|, |b|, |c|, |d| \leq 1$ , but the pairwise products of  $a, b, c, d$  are

not  $\geq 1$ . When the condition  $|a|, |b|, |c|, |d| \leq 1$  on the parameters is dropped, finitely many discrete terms have to be added to the orthogonality relation (4.1).

When we compare the expression for  $\Delta^+(x)$  with the Askey-Wilson weight function we see that Macdonald's polynomials for root system  $BC_1$  coincide, up to a constant factor, with Askey-Wilson polynomials

$$p_l(\cos x; q^{1/2}, -q^{1/2}, ab^{1/2}, -b^{1/2} | q).$$

By Askey & Wilson [1, (4.16), (4.17), (4.20)] the continuous  $q$ -Jacobi polynomials in M. Rahman's notation can be expressed in terms of Askey-Wilson polynomials by

$$\begin{aligned} P_l^{(\alpha, \beta)}(\cos x; q) &= \text{const. } p_l(\cos x; q^{1/2}, -q^{1/2}, q^{\alpha+1/2}, -q^{\beta+1/2} | q) \\ &= \text{const. } p_l(\cos x; q^{\alpha+1/2}, q^{\alpha+3/2}, -q^{\beta+1/2}, -q^{\beta+3/2} | q^2). \end{aligned}$$

Thus, if we put  $a := q^\alpha, b := q^{2\beta}$ , then Macdonald's polynomials for root system  $BC_1$  coincide, up to a constant factor, with continuous  $q$ -Jacobi polynomials  $P_l^{(\alpha+\beta-1/2, \beta-1/2)}(\cos x; q)$ . This observation was already made by Macdonald [12, §9].

If  $n = 1$  then, with  $\sigma := 1$ ,

$$\Phi_\sigma(x) = \frac{(1 - a b^{1/2} e^{ix})(1 + b^{1/2} e^{ix})}{1 - e^{2ix}}.$$

Thus, if we write

$$R_l(e^{ix}) := P_l(x)$$

then Theorem 2.3 yields

$$\Phi_\sigma(-x) R_l(q^{-1} e^{ix}) + \Phi_\sigma(x) R_l(q e^{ix}) = (abq^l + q^{-l}) R_l(e^{ix}).$$

Compare this with Askey & Wilson [1, (5.7), (5.8), (5.9)]:

$$\begin{aligned} A(-x) (R_l(q^{-1} e^{ix}) - R_l(e^{ix})) + A(x) (R_l(q e^{ix}) - R_l(e^{ix})) \\ = -(1 - q^{-l})(1 - q^{l-1}abcd) R_l(e^{ix}), \end{aligned}$$

where

$$R_l(e^{ix}) := \text{const. } p_l(\cos x; a, b, c, d | q)$$

and

$$A(x) := \frac{(1 - ae^{ix})(1 - be^{ix})(1 - ce^{ix})(1 - de^{ix})}{(1 - e^{2ix})(1 - qe^{2ix})}.$$

If  $c = q^{1/2}, d = -q^{1/2}$  (the continuous  $q$ -Jacobi case) then

$$A(x) = \frac{(1 - ae^{ix})(1 - be^{ix})}{1 - e^{2ix}}$$

and

$$A(x) + A(-x) = 1 - ab,$$

so

$$A(-x) R_l(q^{-1} e^{ix}) + A(x) R_l(q e^{ix}) = (q^{-l} - q^l ab) R_l(e^{ix}).$$

Thus Macdonald's difference equation for  $P_l$  in case  $R = BC_1$  coincides with the continuous  $q$ -Jacobi case of the difference equation for Askey-Wilson polynomials.

5. Askey-Wilson polynomials for root system  $BC_n$

We use the notation of §2 and §3. Let

$$\begin{aligned} R_1^+ &:= \{2\varepsilon_j\}_{j=1,\dots,n}, & R_2^+ &:= \{\varepsilon_i \pm \varepsilon_j\}_{1 \leq i < j \leq n}, \\ R_1 &:= R_1^+ \cup (-R_1^+), & R_2 &:= R_2^+ \cup (-R_2^+), \\ R_\ell^+ &:= R_1^+ \cup R_2^+, & R_\ell &:= R_1 \cup R_2 = R_\ell^+ \cup (-R_\ell^+). \end{aligned}$$

Let  $R$  be the root system of type  $BC_n$  of §3. Then  $R_\ell = \{\alpha \in R \mid 2\alpha \notin R\}$ , a root system of type  $C_n$  in  $V$  with subsystems  $R_1$  of type  $nA_1$  and  $R_2$  of type  $D_n$ . (The subscript  $\ell$  stands for 'long'.) Let  $W$  be the Weyl group of  $R_\ell$ . It is a semidirect product of the group of permutations of the coordinates and the group of sign changes of the coordinates. Let  $\rho, \rho_1, \rho_2$  denote half the sum of the positive roots of  $R_\ell, R_1, R_2$ , respectively. Then  $\rho = \rho_1 + \rho_2$  and

$$\rho_1 = \varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n, \quad \rho_2 = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \dots + \varepsilon_{n-1}.$$

Let  $P, P^+$  and the partial order  $\leq$  be as in §3. Write  $\varepsilon(w) := \det(w)$  ( $w \in W$ ). Let  $A^{W,\varepsilon}$  consist of all  $f \in A$  such that  $wf = \varepsilon(w)f$  ( $w \in W$ ). Write

$$J_\lambda := \sum_{w \in W} \varepsilon(w) e^{w\lambda}, \quad \lambda \in P.$$

The  $J_{\lambda+\rho}$  ( $\lambda \in P^+$ ) form a basis of  $A^{W,\varepsilon}$ . In particular, put

$$\delta := J_\rho = \prod_{\alpha \in R_\ell^+} (e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}) = e^\rho \prod_{\alpha \in R_\ell^+} (1 - e^{-\alpha})$$

and

$$\chi_\lambda := \delta^{-1} J_{\lambda+\rho}, \quad \lambda \in P.$$

The  $\chi_\lambda$  ( $\lambda \in P^+$ ) are in  $A^W$  and form a basis of  $A^W$ . We have

$$\chi_\lambda = m_\lambda + \sum_{\mu \in P^+; \mu < \lambda} a_{\lambda,\mu} m_\mu, \quad \lambda \in P^+,$$

for certain complex  $a_{\lambda,\mu}$ .

Fix  $q \in (0, 1)$  and  $a, b, c, d, t \in \mathbb{C}$ . Let

$$(5.1) \quad \Delta^+ := \prod_{\alpha \in R_1^+} \frac{(e^\alpha; q)_\infty}{(ae^{\frac{1}{2}\alpha}, be^{\frac{1}{2}\alpha}, ce^{\frac{1}{2}\alpha}, de^{\frac{1}{2}\alpha}; q)_\infty} \prod_{\alpha \in R_2^+} \frac{(e^\alpha; q)_\infty}{(te^\alpha; q)_\infty}$$

and

$$(5.2) \quad \Delta(x) := \Delta^+(x) \Delta^+(-x).$$

We are now ready to introduce Askey-Wilson polynomials for root system  $BC_n$ .



DEFINITION 5.1. Assume  $a, b, c, d$  are real, or if complex, appear in conjugate pairs, and that  $|a|, |b|, |c|, |d| \leq 1$ , but the pairwise products of  $a, b, c, d$  are not  $\geq 1$ . Assume  $-1 < t < 1$ . Let  $T := [-\pi, \pi]^n \subset V$ . For  $\lambda \in P^+$  define  $P_\lambda \in A^W$  by the two conditions

- (i)  $P_\lambda = m_\lambda + \sum_{\mu \in P^+; \mu < \lambda} u_{\lambda, \mu} m_\mu$  for certain coefficients  $u_{\lambda, \mu}$ ;
- (ii)  $\int_T P_\lambda(x) m_\mu(x) \Delta(x) dx = 0$  for  $\mu \in P^+, \mu < \lambda$ .

We will generalize the case  $R = BC_n$  of Theorem 2.2 by showing that the  $P_\lambda$  are orthogonal on  $T$  with respect to the weight function  $\Delta$ . The proof will be based on two lemmas, the first one giving the action of a suitable difference operator on the  $m_\lambda$ , and the second one showing self-adjointness of this operator with respect to  $\Delta$  on  $T$ , when acting on  $A^W$ .

Let  $\sigma := \varepsilon_1$ , similarly as in §3 for the pair  $(BC_n, B_n)$ . Define  $\Phi_\sigma$  and  $D_\sigma$  as in §2, with  $\Delta^+$  being given by (5.1). Thus

$$(5.3) \quad \Phi_\sigma := \frac{T_\sigma \Delta^+}{\Delta^+},$$

and

$$(5.4) \quad \begin{aligned} D_\sigma f &:= |W_\sigma|^{-1} \sum_{w \in W} w(\Phi_\sigma(T_\sigma f - f)) \\ &= |W_\sigma|^{-1} \sum_{w \in W} (w\Phi_\sigma)(T_w \sigma f - f), \quad f \in A^W. \end{aligned}$$

LEMMA 5.2.

$$D_\sigma m_\lambda = \sum_{\mu \in P^+; \mu \leq \lambda} a_{\lambda, \mu} m_\mu$$

with

$$(5.5) \quad a_{\lambda, \lambda} = \sum_{j=1}^n (q^{-1}abcdt^{2n-j-1}(q^{\lambda_j} - 1) + t^{j-1}(q^{-\lambda_j} - 1)).$$

Here  $a, b, c, d, t$  may be arbitrarily complex.

PROOF. It will be convenient to replace  $a, b, c, d$  in the expression (5.1) for  $\Delta^+$  by  $a, -b, q^{\frac{1}{2}}c, -q^{\frac{1}{2}}d$ , respectively. Thus

$$(5.6) \quad \Delta^+ := \prod_{\alpha \in R_1^+} \frac{(e^\alpha; q)_\infty}{(ae^{\frac{1}{2}\alpha}, -be^{\frac{1}{2}\alpha}, q^{\frac{1}{2}}ce^{\frac{1}{2}\alpha}, -q^{\frac{1}{2}}de^{\frac{1}{2}\alpha}; q)_\infty} \prod_{\alpha \in R_2^+} \frac{(e^\alpha; q)_\infty}{(te^\alpha; q)_\infty}.$$

By substitution of (5.6) in (5.3) we obtain

$$\begin{aligned} \Phi_\sigma &= \frac{(1 - ae^{\varepsilon_1})(1 + be^{\varepsilon_1})(1 - q^{\frac{1}{2}}ce^{\varepsilon_1})(1 + q^{\frac{1}{2}}de^{\varepsilon_1})}{(1 - e^{2\varepsilon_1})(1 - qe^{2\varepsilon_1})} \prod_{\alpha=\varepsilon_1 \pm \varepsilon_i; i=2, \dots, n} \frac{1 - te^\alpha}{1 - e^\alpha} \\ &= abcd t^{2(n-1)} \frac{(1 - a^{-1}e^{-\varepsilon_1})(1 + b^{-1}e^{-\varepsilon_1})}{(1 - e^{-2\varepsilon_1})} \\ &\quad \times \frac{(1 - q^{-\frac{1}{2}}c^{-1}e^{-\varepsilon_1})(1 + q^{-\frac{1}{2}}d^{-1}e^{-\varepsilon_1})}{(1 - q^{-1}e^{-2\varepsilon_1})} \prod_{\alpha=\varepsilon_1 \pm \varepsilon_i; i=2, \dots, n} \frac{1 - t^{-1}e^{-\alpha}}{1 - e^{-\alpha}} \\ &= abcd t^{2(n-1)} \prod_{\alpha \in R_2^+} \frac{1 - t^{-(\sigma, \alpha)}e^{-\alpha}}{1 - e^{-\alpha}} \prod_{j=1}^n \left[ \frac{(1 - a^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})(1 + b^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})}{(1 - e^{-2\varepsilon_j})} \right. \\ &\quad \left. \times \frac{(1 - q^{-\frac{1}{2}}c^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})(1 + q^{-\frac{1}{2}}d^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})}{(1 - q^{-1}e^{-2\varepsilon_j})} \right]. \end{aligned}$$

Hence

$$\Phi_\sigma = \delta^{-1} \delta_q^{-1} \Psi_\sigma,$$

where

$$\begin{aligned} (5.7) \quad \Psi_\sigma &:= (abcd)^{(\sigma, \rho_1)} t^{(\sigma, 2\rho_2)} e^{\rho+2\rho_1} \prod_{\alpha \in R_2^+} (1 - t^{-(\sigma, \alpha)}e^{-\alpha}) \\ &\quad \times \prod_{j=1}^n [(1 - qe^{-2\varepsilon_j})(1 - a^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})(1 + b^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j}) \\ &\quad \times (1 - q^{-\frac{1}{2}}c^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})(1 + q^{-\frac{1}{2}}d^{-(\sigma, \varepsilon_j)}e^{-\varepsilon_j})] \end{aligned}$$

and  $\delta_q$  is the following element of  $A^W$ :

$$\begin{aligned} (5.8) \quad \delta_q &:= \prod_{j=1}^n (q^{-\frac{1}{2}}e^{\varepsilon_j} - q^{\frac{1}{2}}e^{-\varepsilon_j})(q^{\frac{1}{2}}e^{\varepsilon_j} - q^{-\frac{1}{2}}e^{-\varepsilon_j}) \\ &= e^{2\rho_1} \prod_{j=1}^n (1 - qe^{-2\varepsilon_j})(1 - q^{-1}e^{-2\varepsilon_j}). \end{aligned}$$

If  $E \subset (\frac{1}{2}R_1) \cup R_2$  then write

$$\|E\| := \sum_{\alpha \in E} \alpha.$$

Expansion of (5.7) yields

$$(5.9) \quad \Psi_\sigma = \sum_{E_0, \dots, E_4 \subset \frac{1}{2}R_1^+} \sum_{F \subset R_2^+} c_{E_0, \dots, E_4, F} e^{\rho+2\rho_1-2\|E_0\|-\|E_1\|-\dots-\|E_4\|-\|F\|},$$

where

$$\begin{aligned} (5.10) \quad c_{E_0, \dots, E_4, F} &:= (-1)^{|E_0|+|E_1|+|E_3|+|F|} q^{|E_0|-\frac{1}{2}|E_3|-\frac{1}{2}|E_4|} \\ &\quad \times a^{(\sigma, \rho_1-\|E_1\|)} b^{(\sigma, \rho_1-\|E_2\|)} c^{(\sigma, \rho_1-\|E_3\|)} d^{(\sigma, \rho_1-\|E_4\|)} t^{(\sigma, 2\rho_2-\|F\|)}. \end{aligned}$$

Now we can rewrite (5.4) as

$$D_\sigma f = \delta^{-1} \delta_q^{-1} \tilde{D}_\sigma f$$

with

$$(5.11) \quad \tilde{D}_\sigma f := |W_\sigma|^{-1} \sum_{w \in W} \varepsilon(w) (w \Psi_\sigma) (T_{w\sigma} f - f).$$

Consider (5.11) with  $f := m_\lambda$  ( $\lambda \in P^+$ ) and substitute (5.9). Then

$$\begin{aligned} \tilde{D}_\sigma m_\lambda &= |W_\sigma|^{-1} |W_\lambda|^{-1} \sum_{w_1, w_2 \in W} \sum_{E_0, \dots, E_4 \subset \frac{1}{2} R_1^+} \sum_{F \subset R_2^+} c_{E_0, \dots, E_4, F} \varepsilon(w_1) \\ &\quad \times (q^{(w_1 \sigma, w_2 \lambda)} - 1) e^{w_1(\rho + 2\rho_1 - 2\|E_0\| - \|E_1\| - \dots - \|E_4\| - \|F\|) + w_2 \lambda}. \end{aligned}$$

Put  $w_2 = w_1 w$ . Then

$$(5.12) \quad \begin{aligned} \tilde{D}_\sigma m_\lambda &= |W_\sigma|^{-1} |W_\lambda|^{-1} \sum_{w \in W} \sum_{E_0, \dots, E_4 \subset \frac{1}{2} R_1^+} \sum_{F \subset R_2^+} c_{E_0, \dots, E_4, F} (q^{(\sigma, w\lambda)} - 1) \\ &\quad \times J_{w\lambda + \rho + 2\rho_1 - 2\|E_0\| - \|E_1\| - \dots - \|E_4\| - \|F\|}. \end{aligned}$$

Hence  $\tilde{D}_\sigma m_\lambda \in A^{W, \varepsilon}$ . Now the  $J$ -function in (5.12) is either 0 or  $\varepsilon(w') \delta \chi_\nu$ , where  $w' \in W$ ,  $\nu \in P^+$  and

$$w'(\nu + \rho) = w\lambda + \rho + 2\rho_1 - 2\|E_0\| - \|E_1\| - \dots - \|E_4\| - \|F\|,$$

so that

$$(5.13) \quad \begin{aligned} \nu + \rho &= (w')^{-1} w\lambda \\ &\quad + (w')^{-1} (3\rho_1 - 2\|E_0\| - \|E_1\| - \dots - \|E_4\|) + (w')^{-1} (\rho_2 - \|F\|). \end{aligned}$$

Now

$$(5.14) \quad (w')^{-1} (3\rho_1 - 2\|E_0\| - \|E_1\| - \dots - \|E_4\|) = (w')^{-1} \sum_{j=1}^n k_j \varepsilon_j = \sum_{j=1}^n k'_j \varepsilon_j \leq 3\rho_1$$

with  $k_j, k'_j \in \{-3, -2, -1, 0, 1, 2, 3\}$ ,

$$(5.15) \quad (w')^{-1} (\rho_2 - \|F\|) = (w')^{-1} \sum_{\alpha \in R_2^+} k_\alpha \alpha = \sum_{\alpha \in R_2^+} k'_\alpha \alpha \leq \rho_2$$

with  $k_\alpha, k'_\alpha = \pm \frac{1}{2}$ , and

$$(5.16) \quad (w')^{-1} w\lambda \leq \lambda.$$

Substitution of (5.14), (5.15), (5.16) in (5.13) yields

$$(5.17) \quad \nu + \rho \leq \lambda + 3\rho_1 + \rho_2 = \lambda + \rho + 2\rho_1.$$

Hence

$$(5.18) \quad \tilde{D}_\sigma m_\lambda = \sum_{\nu \in P^+; \nu \leq \lambda + 2\rho_1} b_\nu J_{\nu + \rho}$$

for certain coefficients  $b_\nu$ .

In order to compute  $b_{\lambda+2\rho_1}$  observe that equality in (5.17) holds iff equality holds in (5.14), (5.15), (5.16), i.e., iff  $(w')^{-1}w \in W_\lambda$  and

$$(5.19) \quad \|E_0\| = \|E_1\| = \cdots = \|E_4\| = \frac{1}{2}(\rho_1 - w'\rho_1),$$

$$(5.20) \quad \|F\| = \rho_2 - w'\rho_2.$$

Hence, by (5.12),

$$(5.21) \quad b_{\lambda+2\rho_1} = |W_\sigma|^{-1} |W_\lambda|^{-1} \sum_{w, w' \in W; (w')^{-1}w \in W_\lambda} \varepsilon(w') c_{E_0, \dots, E_4, F} (q^{(\sigma, w\lambda)} - 1),$$

where  $E_0, \dots, E_4, F$  are determined by (5.19), (5.20). It follows from (5.19), (5.20) that

$$2\|E_0\| + \|F\| = \rho - w'\rho, \quad \text{hence} \quad (-1)^{|E_0|+|F|} = \varepsilon(w').$$

Substitution of (5.19), (5.20) into (5.10) now yields:

$$c_{E_0, \dots, E_4, F} = \varepsilon(w') (abcd)^{\frac{1}{2}(1 + ((w')^{-1}\sigma, \rho_1))} t^{n-1 + ((w')^{-1}\sigma, \rho_2)}.$$

When we substitute this last expression into (5.21) then we obtain

$$\begin{aligned} b_{\lambda+2\rho_1} &= |W_\sigma|^{-1} |W_\lambda|^{-1} \\ &\times \sum_{w, w' \in W; (w')^{-1}w \in W_\lambda} (abcd)^{\frac{1}{2}(1 + ((w')^{-1}\sigma, \rho_1))} t^{n-1 + ((w')^{-1}\sigma, \rho_2)} (q^{((w')^{-1}\sigma, \lambda)} - 1) \\ &= |W_\sigma|^{-1} \sum_{w \in W} (abcd)^{\frac{1}{2}(1 + (w\sigma, \rho_1))} t^{n-1 + (w\sigma, \rho_2)} (q^{(w\sigma, \lambda)} - 1). \end{aligned}$$

Hence

$$(5.22) \quad b_{\lambda+2\rho_1} = \sum_{j=1}^n \sum_{\varepsilon = \pm 1} (abcd)^{\frac{1}{2}(1 + \varepsilon)} t^{n-1 + \varepsilon(n-j)} (q^{\varepsilon\lambda_j} - 1),$$

which is (5.5), when we take in account the replacement made for  $a, b, c, d$ .

Next we show that, for  $f \in A^W$ ,  $\tilde{D}_\sigma f$  given by (5.11) is divisible by  $\delta_q$ . In view of (5.8) this will follow if we can show that, for each  $w \in W$ ,  $(w\Psi_\sigma)(T_{w\sigma}f - f)$  is divisible by the  $4n$  prime factors  $1 \pm q^{\pm \frac{1}{2}} e^{-\varepsilon_j}$ . By (5.7), all but the two factors  $1 \pm q^{-\frac{1}{2}} e^{-w\sigma}$  are divisors of  $w\Psi_\sigma$ . We will show that these two factors are divisors of  $T_{w\sigma}f - f$ . Write  $f$  as a Laurent polynomial  $F(e^{\varepsilon_1}, \dots, e^{\varepsilon_n})$ , invariant under the transformations  $e^{\varepsilon_j} \mapsto e^{-\varepsilon_j}$ . If  $w\sigma = \varepsilon_j$  then

$$T_{w\sigma}f - f = F(e^{\varepsilon_1}, \dots, qe^{\varepsilon_j}, \dots, e^{\varepsilon_n}) - F(e^{\varepsilon_1}, \dots, e^{\varepsilon_j}, \dots, e^{\varepsilon_n})$$

becomes 0 for  $e^{\epsilon_j} = \pm q^{-\frac{1}{2}}$ , hence it is divisible by  $1 \pm q^{-\frac{1}{2}}e^{-\epsilon_j}$ . A similar argument is valid for  $w\sigma = -\epsilon_j$ .

By (5.18),

$$(5.23) \quad \delta^{-1} \tilde{D}_\sigma m_\lambda = \sum_{\nu \in P^+; \nu \leq \lambda + 2\rho_1} c_\nu m_\nu$$

for certain coefficients  $c_\nu$ , with  $c_{\lambda+2\rho_1} = b_{\lambda+2\rho_1}$  given by (5.22). Also,  $\delta^{-1} \tilde{D}_\sigma m_\lambda$  will still be divisible by  $\delta_q$ . By (5.8),  $\delta_q \in A^W$  with highest term  $m_{2\rho_1}$ . Hence  $D_\sigma m_\lambda = \delta^{-1} \delta_q^{-1} \tilde{D}_\sigma m_\lambda$  will be in  $A^W$  with highest term  $b_{\lambda+2\rho_1} m_\lambda$ .  $\square$

We have

$$\Delta = \prod_{\alpha \in R_1} \frac{(e^\alpha; q)_\infty}{(ae^{\frac{1}{2}\alpha}, be^{\frac{1}{2}\alpha}, ce^{\frac{1}{2}\alpha}, de^{\frac{1}{2}\alpha}; q)_\infty} \prod_{\alpha \in R_2} \frac{(e^\alpha; q)_\infty}{(te^\alpha; q)_\infty}.$$

Hence

$$\Delta(x) = (w\Delta)(x) = (w\Delta^+)(x) (w\Delta^+)(-x), \quad w \in W.$$

LEMMA 5.3. *With the assumptions of Definition 5.1 we have*

$$(5.24) \quad \int_T (D_\sigma f)(x) g(x) \Delta(x) dx = \int_T f(x) (D_\sigma g)(x) \Delta(x) dx, \quad f, g \in A^W.$$

PROOF. Since  $-\text{id} \in W$ ,  $f(x) = f(-x)$  and  $g(x) = g(-x)$ . By (5.4) and (5.3), formula (5.24) can be equivalently written as

$$(5.25) \quad \begin{aligned} & \sum_{w \in W} \int_T (T_{w\sigma}(w\Delta^+))(x) ((T_{w\sigma} f)(x) - f(x)) (w\Delta^+)(-x) g(-x) dx \\ &= \sum_{w \in W} \int_T (w\Delta^+)(x) f(x) (T_{w\sigma}(w\Delta^+))(-x) ((T_{w\sigma} g)(-x) - g(-x)) dx. \end{aligned}$$

Since  $T$ ,  $f$  and  $g$  are  $W$ -invariant, formula (5.25) will be implied by the two identities

$$\int_T (\Delta^+ f)(x - i(\log q)\sigma) (\Delta^+ g)(-x) dx = \int_T (\Delta^+ f)(x) (\Delta^+ g)(-x - i(\log q)\sigma) dx$$

and

$$\sum_{w \in W} (T_\sigma \Delta^+)(w^{-1}x) \Delta^+(-w^{-1}x) = \sum_{w \in W} \Delta^+(w^{-1}x) (T_\sigma \Delta^+)(-w^{-1}x).$$

The second identity is obvious, since  $-\text{id} \in W$ . For the first identity observe that the integral

$$\int_C (\Delta^+ f)(z - i(\log q), x_2, \dots, x_n) (\Delta^+ g)(-z, -x_2, \dots, -x_n) dz$$

over the contour

$$C = [-\pi, \pi] \cup [\pi, \pi + i \log q] \cup [\pi + i \log q, -\pi + i \log q] \cup [-\pi + i \log q, -\pi]$$

vanishes by Cauchy's theorem. (By the assumptions on  $a, b, c, d, t$  there are no singularities inside the contour.) Now the result follows, since  $\Delta^+ f$  and  $\Delta^+ g$  are invariant under translations by  $2\pi\sigma$ .  $\square$

It follows now immediately from Lemmas 5.2 and 5.3 that:

**THEOREM 5.4.**  $D_\sigma P_\lambda = a_{\lambda,\lambda} P_\lambda$  with  $a_{\lambda,\lambda}$  given by (5.5).

Now we are ready for the main theorem.

**THEOREM 5.5.** *If  $\lambda, \mu \in P^+$ ,  $\lambda \neq \mu$ , then*

$$\int_T P_\lambda(x) P_\mu(x) \Delta(x) dx = 0.$$

**PROOF.** All integrals

$$\int_T m_\lambda(x) m_\mu(x) \Delta(x) dx$$

are continuous in  $a, b, c, d, t$ . Hence the coefficients  $u_{\lambda,\mu}$  in Definition 5.1 are continuous in  $a, b, c, d, t$ . This implies that

$$\int_T P_\lambda(x) P_\mu(x) \Delta(x) dx$$

is continuous in  $a, b, c, d, t$ . By Theorem 5.4 and Lemma 5.3,

$$\int_T P_\lambda(x) P_\mu(x) \Delta(x) dx = 0$$

if  $a_{\lambda,\lambda} \neq a_{\mu,\mu}$ . Fix distinct  $\lambda$  and  $\mu$  it follows from (5.5) that, for fixed nonzero  $a, b, c, d$ , the eigenvalues  $a_{\lambda,\lambda}$  and  $a_{\mu,\mu}$  are distinct as polynomials in  $t$ . This implies the orthogonality of  $P_\lambda$  and  $P_\mu$  for  $a, b, c, d, t$  in a dense subset of the parameter domain under consideration. Hence, by continuity, the theorem follows.  $\square$

The method of proof in this last theorem is different from the method used in similar situations by Macdonald [12]. While Macdonald leaves the parameters fixed and shows that equality of eigenvalues for all  $q$  implies (in most cases) equality of weights, the above proof leaves  $q$  fixed and shows that equality of eigenvalues for all parameter values implies equality of weights.

## 6. Discussion of results

**6.1. Special cases.** When we compare (5.1) with (3.1), (3.2) and (3.3), then it is clear that Askey-Wilson polynomials for root system  $BC_n$  with  $a, b, c, d, t$  replaced by  $q^{\frac{1}{2}}, -q^{\frac{1}{2}}, ab^{\frac{1}{2}}, -b^{\frac{1}{2}}, t$  become Macdonald's polynomials for the pair  $(BC_n, B_n)$ . The operator  $D_\sigma$  given by (5.4) then specializes to the operator for which Theorem 2.3 is valid in case  $(BC_n, B_n)$ , and the eigenvalue (5.5) specializes to the eigenvalue in Theorem 2.3, cf. (3.6). We can also work then with  $E_\sigma$  instead of  $D_\sigma$ .

Next, when we compare (5.1) with (3.1), (3.4) and (3.5) then it is clear that our polynomials with  $a, b, c, d, t, q$  replaced by  $ab^{\frac{1}{2}}, qab^{\frac{1}{2}}, -b^{\frac{1}{2}}, -qb^{\frac{1}{2}}, t, q^2$  become Macdonald's polynomials for the pair  $(BC_n, C_n)$ . The operator  $D_\sigma$  given by (5.4) then becomes the operator  $D_{2\epsilon_1}$  for the pair  $(BC_n, C_n)$ . Theorem 2.3 does not say anything about eigenfunctions of this operator, but Theorem 5.4 implies that Macdonald's polynomials for the pair  $(BC_n, C_n)$  are eigenfunctions of  $D_{2\epsilon_1}$ . This corresponds nicely with the cases  $E_8, F_4, G_2$  of Theorem 2.3, where  $\langle \sigma, \alpha_* \rangle$  takes values 0, 1, 2 as  $\alpha$  runs through  $R^+$  and we have to work with  $D_\sigma$  instead of  $E_\sigma$ . It would be interesting to consider if  $P_\lambda$  might also be eigenfunction of  $D_\sigma$  for other "quasi-minuscule"  $\sigma$ .

Comparison of (5.1) and (4.1) makes it evident that the  $BC_n$  Askey-Wilson polynomials reduce to the one-variable Askey-Wilson polynomials for  $n = 1$ .

**6.2. A Selberg-type integral and a conjectured quadratic norm.** Let  $\Delta$  be given by (5.2) and (5.1) and let the parameters satisfy the inequalities of Definition 5.1. Gustafson [3, (2)] evaluated the Selberg type integral

$$\frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} \Delta(x) dx = 2^n n! \prod_{j=1}^n \frac{(t, t^{n+j-2}abcd; q)_\infty}{(t^j, q, abt^{j-1}, act^{j-1}, \dots, cdt^{j-1}; q)_\infty}.$$

On the other hand, Macdonald [12, (12.6)] conjectured an explicit expression for the quadratic norm  $\langle P_\lambda, P_\lambda \rangle$  for polynomials  $P_\lambda$  associated with any admissible pair  $(R, S)$ . It can be shown that Macdonald's conjecture in case  $\lambda = 0$  and  $(R, S) = (BC_n, B_n)$  coincides with Gustafson's formula for  $(a, b, c, d) = (q^{\frac{1}{2}}, -q^{\frac{1}{2}}, ab^{\frac{1}{2}}, -b^{\frac{1}{2}})$ . In October 1991, when prof. Macdonald was visiting The Netherlands, first the author has given a conjectured expression for  $\langle P_\lambda, P_\lambda \rangle / \langle 1, 1 \rangle$ , where  $P_\lambda$  is an Askey-Wilson polynomial for root system  $BC_n$ , and next Macdonald [13] has rewritten this as a conjectured expression for  $\langle P_\lambda, P_\lambda \rangle$ . On the same occasion, Macdonald [13] has also extended his other conjectures in [12, §12] to the  $BC_n$  Askey-Wilson case.

**6.3. The Askey-Wilson hierarchy for  $BC_n$ .** It is very probable that all specializations and limit cases of one-variable Askey-Wilson polynomials have their analogues in the case of  $BC_n$ . Someone should certainly write down the orthogonality relations and difference operators with explicit eigenvalues for all these specializations. In some cases these explicit formulas may be rigorously proved by straightforward limit transition from the general Askey-Wilson case. In other cases, the limit transition may only give a formal proof and, for a rigorous derivation, the proofs of the present paper will have to be imitated.

$q$ -Racah-type polynomials for root system  $BC_n$  should also be obtained. Here analytic continuation from the  $BC_n$  Askey-Wilson polynomials will be needed and residues, possibly higher dimensional, will have to be taken. Similar problems will arise when the condition  $|a|, |b|, |c|, |d| \leq 1$  is dropped in Definition 5.1. In the corresponding one-variable case discrete terms are then added to the orthogonality relations.

**6.4. Quantum group interpretations.** It is known from work by Koornwinder [9], [10], Koelink [8] and Noumi & Mimachi [15], [16] that one-variable Askey-Wilson polynomials have an interpretation on the quantum group  $SU_q(2)$ . Noumi [14] announces an interpretation of Macdonald's polynomials for root system  $A_{n-1}$  as zonal spherical functions on the quantum analogues of the homogeneous spaces  $GL(n)/SO(n)$  and  $GL(2n)/Sp(2n)$ . According to Noumi, this was already done for the quantum analogue of  $SL(3)/SO(3)$  by Ueno & Takebayashi. It would be interesting to find quantum group interpretations of Macdonald's polynomials in case of all root systems, and also of the  $BC_n$  Askey-Wilson polynomials considered in the present paper.

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